

ON A DIFFERENTIAL INCLUSION RELATED TO THE BORN-INFELD EQUATIONS

STEFAN MÜLLER¹ AND MARIAPIA PALOMBARO²

¹ Hausdorff Center for Mathematics & Institute for Applied Mathematics, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany. Email: sm@hcm.uni-bonn.de

² SISSA, Via Bonomea 265, 34136 Trieste, Italy. Email: palombar@sissa.it

ABSTRACT. We adapt the well-known method of convex integration to the setting of divergence free fields in order to find solenoidal maps taking values in a prescribed set of matrices and with a prescribed average. As an application we study a partial differential relation arising in the study of the Born-Infeld equations.

Key words: solenoidal fields, \mathcal{A} -quasiconvexity, convex integration, Born-Infeld equations.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $K \subset \mathbb{M}^{m \times n}$ be a set of $m \times n$ real matrices. We study the problem of whether there exist solutions to the differential inclusion

$$(1.1) \quad \begin{cases} \operatorname{Div} V = 0 & \text{in } \mathcal{D}'(\Omega; \mathbb{R}^m), \\ V \in K & \text{a.e. in } \Omega, \\ \int_{\Omega} V = F, \end{cases}$$

for some given $F \in \mathbb{M}^{m \times n}$. Our interest in this question arises, in particular, from applications to the study of the Born-Infeld equations. In fact, we will consider a special case of (1.1), when $m = 2$, $n = 3$, and the set K is related to the so-called Born-Infeld manifold. Further applications of solenoidal differential inclusions can be found in the study of composite materials, as well as linear elasticity and fluid mechanics (see, e.g., [5], [11], [2], [3]). More generally, problem (1.1) falls into the framework of \mathcal{A} -quasiconvexity, where the differential constraint on the function V is replaced by more general ones (see, e.g., [4]).

Our approach to (1.1) is based on the extension to the div-free setting of the well-known method of convex integration. The latter has been introduced and developed by Gromov to solve partial differential relations in connection with geometric problems. In this context the problem is to find gradients, i.e. curl-free fields, that take values in a prescribed set of matrices, namely, solutions of the partial differential inclusion

$$\nabla u \in K.$$

We refer to Gromov's treatise [6] for a detailed exposition and further references concerning the existence of C^1 solutions. Müller and Šverák [10] have subsequently adapted this method for constructing regularity counterexamples for elliptic systems. For an overview on the techniques and results available in the Lipschitz setting we refer the reader to [8]. Let us also mention that extensions of the convex integration methods to constraints different from curl can be found in [3].

The purpose of this paper is to extend this method to the div-free setting in order to solve (1.1). This is the content of Section 3. Our main result, Theorem 3.8, states that problem (1.1) admits a solution whenever K can be "approximated" in the sense of Definition 3.7 and F lies in the interior of some appropriate hull of K .

In Section 4 we specialize the results obtained in Section 3 to the case of a partial differential relation arising in connection with the Born-Infeld equations. Let us briefly introduce the problem. The Born-Infeld system is a non-linear version of Maxwell equations which can be written as

$$(1.2) \quad \begin{aligned} \partial_t D + \operatorname{curl} \left(\frac{-B + D \wedge P}{h} \right) &= \partial_t B + \operatorname{curl} \left(\frac{D + B \wedge P}{h} \right) = 0, \\ \operatorname{div} D &= \operatorname{div} B = 0, \end{aligned}$$

where the unknowns $D, B : \Omega \times [0, T] \subset \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$ are divergence free fields and the functions P and h are defined by the following relations

$$(1.3) \quad P = D \wedge B, \quad h = \sqrt{1 + |B|^2 + |D|^2 + |P|^2}.$$

Relations (1.3) define a six-dimensional manifold in \mathbb{R}^{10} , that we call the BI-manifold and denote by \mathcal{M} . We refer to Brenier [1] for the mathematical analysis and many further references on the Born-Infeld equations (1.2). Here we only give a brief account of those arguments of [1] which give rise to the question addressed in this paper. The starting point is to observe that if (D, B) are smooth solutions of (1.2), then P and h satisfy the additional conservation laws

$$(1.4) \quad \begin{aligned} \partial_t h + \operatorname{div} P &= 0, \\ \partial_t P + \operatorname{Div} \left(\frac{P \otimes P - B \otimes B - D \otimes D}{h} \right) &= \operatorname{curl} \left(\frac{1}{h} \right). \end{aligned}$$

This suggests to lift the 6×6 system (1.2) to a 10×10 system of conservation laws by adding equations (1.4), regardless condition (1.3). Namely, one regards P and h as additional unknowns. The resulting augmented system (1.2)-(1.4) enjoys remarkable properties which allow for an easier analysis than the original system (1.2) (see [1] for more precise details). Of course, among all solutions of the augmented system, only those with initial conditions valued in the BI-manifold genuinely correspond to the original system (1.2), as they continue to satisfy (1.3) also for positive times. A natural question then, is to understand which initial conditions can be weakly approximated by initial conditions valued in the BI-manifold \mathcal{M} . This is exactly the object of Section 4. Specifically, we investigate the following question. Given a sequence $\{V_j\} = \{(D_j, B_j, P_j, h_j)\} \subset L^p(\Omega; \mathbb{R}^{10})$, such that

$$(1.5) \quad \operatorname{div} D_j = \operatorname{div} B_j = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

$$(1.6) \quad V_j \in \mathcal{M} \quad \text{a.e. in } \Omega,$$

$$(1.7) \quad V_j \rightharpoonup V \quad \text{in } L^p(\Omega),$$

what is the range of the limit function V ? It is known that V takes values in convex hull \mathcal{M}^c of \mathcal{M} . Conversely, given $V = (D, B, P, h) \in L^p(\Omega; \mathbb{R}^{10})$ with $V \in \mathcal{M}^c$ a.e., and $\operatorname{div} D = \operatorname{div} B = 0$ in $\mathcal{D}'(\Omega)$, does there exist a sequence $\{V_j\}$ satisfying (1.5)-(1.7)? The answer to this question is already present, to some extent, in [1], though only mentioned. In Section 4.1 we clarify it further by setting the problem in the framework of Young measures. We also remark that, by this approach, one can only obtain a sequence which satisfies (1.5)-(1.6) approximately. Namely, either we have

$$\operatorname{div} D_j = \operatorname{div} B_j = 0 \quad \text{and} \quad \operatorname{dist}(V_j, \mathcal{M}) \rightarrow 0 \quad \text{in measure},$$

or

$$\operatorname{div} D_j \rightarrow 0, \operatorname{div} B_j \rightarrow 0 \quad \text{strongly in } W^{-1,p'} \quad \text{and} \quad V_j \in \mathcal{M} \text{ a.e.}$$

In order to obtain a sequence which satisfies both the constraints (1.5)-(1.6) exactly, we will use the method of convex integration developed in the first part of the paper. This is the content of Section 4.2. Specifically, we will prove the following theorem

Theorem 1.1. *Let $F \in \operatorname{Int}(\mathcal{M}^c)$. Then there exists a sequence $\{V_j\} = \{(D_j, B_j, P_j, h_j)\} \subset L^\infty(\Omega; \mathbb{R}^{10})$ such that*

$$\operatorname{div} D_j = \operatorname{div} B_j = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

$$V_j \in \mathcal{M} \quad \text{a.e.},$$

$$V_j \xrightarrow{*} F \quad \text{in } L^\infty - \text{weak}^*.$$

2. NOTATION

For a matrix $A = (A_{ij}) \in \mathbb{M}^{m \times n}$, we denote by A^i the i th column of A , and by A_i the i th row of A . We say that a matrix field $V \in L^1(\Omega; \mathbb{M}^{m \times n})$ is divergence free, and we write $\operatorname{Div} V = 0$ in $\mathcal{D}'(\Omega; \mathbb{R}^m)$, if each row of the matrix field V is divergence free in the distributional sense. We denote by \mathcal{M} the six-dimensional manifold in \mathbb{R}^{10} defined as

$$(2.1) \quad \mathcal{M} := \{(D, B, P, h) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : P = D \wedge B, h = \sqrt{1 + |D|^2 + |B|^2 + |P|^2}\},$$

and by \mathcal{M}^c its convex hull. For the topological interior of \mathcal{M} we write $\operatorname{Int}(\mathcal{M})$. In Section 4.2 we use the identification $\mathbb{R}^{10} \simeq \mathbb{R}_D^3 \times \mathbb{R}_B^3 \times \mathbb{R}_P^3 \times \mathbb{R}_h$, and for any $M = (M_1, \dots, M_{10}) \in \mathbb{R}^{10}$, we write

$$M = (M_D, M_B, M_P, M_h) \in \mathbb{R}_D^3 \times \mathbb{R}_B^3 \times \mathbb{R}_P^3 \times \mathbb{R}_h.$$

3. CONVEX INTEGRATION FOR SOLENOIDAL FIELDS

We adapt the method of convex integration to the div-free setting, by essentially following [9].

We will work with potentials of divergence free fields. Therefore we introduce the differential operator $\mathcal{L} : (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m \rightarrow L^\infty(\Omega; \mathbb{M}^{m \times n})$, defined as

$$(\mathcal{L}(G))_{kj} := \sum_{i=1}^n \frac{\partial G_{ij}^k}{\partial x_i}, \quad 1 \leq k \leq m, 1 \leq j \leq n, \quad G = (G^1, \dots, G^m).$$

Lemma 3.1. *Let $G^k \in W^{1,\infty}(\Omega; \mathbb{M}^{n \times n})$ be matrix fields for $1 \leq k \leq m$, such that the tensor G^k is skew symmetric for every k , i.e., $G_{ij}^k = -G_{ji}^k$. Then the matrix field $\mathcal{L}(G)$ is divergence free.*

Remark 3.2. In the case of three dimensions, i.e. for $n = 3$, the operator \mathcal{L} reduces itself to the *curl* operator. Indeed, in Section 4.2 we will work with solenoidal fields that are images of the *curl* operator.

The next result provides the basic construction that allows one to define a divergence free field whose values lie in a small neighborhood of two values, and whose potential can be chosen to be zero on the boundary.

Lemma 3.3. *Let $A, B \in \mathbb{M}^{m \times n}$ and let $F := \theta A + (1 - \theta)B$ for some $\theta \in (0, 1)$. Assume that $\text{rank}(A - B) \leq n - 1$. Then for each $\delta > 0$, there exists $V \in L^\infty(\Omega; \mathbb{M}^{m \times n})$ such that*

$$\begin{aligned} V &= \mathcal{L}(G) + F \text{ with } G \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m \text{ and piecewise linear,} \\ \|G\|_{L^\infty(\Omega)} &< \delta, \\ G|_{\partial\Omega} &= 0, \\ \text{dist}(V, \{A, B\}) &< \delta. \end{aligned}$$

Proof. Without loss of generality we may assume that $(A - B)e_n = 0$, and $F = 0$, so that we can write $A = (1 - \theta)(A - B)$ and $B = -\theta(A - B)$. If not, we can replace A and B by $A - F$ and $B - F$ respectively. We first construct a solution for a special domain Ω_ε and then we will complete the proof by an application of the Vitali covering theorem. Let $\Omega_\varepsilon := (-1, 1)^{n-1} \times (0, \varepsilon)$ and let $\chi : \Omega_\varepsilon \rightarrow \{0, 1\}$ be the characteristic function of the set $(-1, 1)^{n-1} \times (0, \varepsilon\theta)$:

$$\chi(x) = \begin{cases} 1 & \text{if } 0 \leq x_n \leq \varepsilon\theta, \\ 0 & \text{if } \varepsilon\theta < x_n \leq \varepsilon. \end{cases}$$

We then define $U := \chi A + (1 - \chi)B$ and remark that U is divergence free, since $(A - B)e_n = 0$. We seek a potential P of U . For each $k = 1, \dots, m$, and $j = 1, \dots, n$, let

$$\begin{aligned} P_{nj}^k(x) &= \begin{cases} A_{kj}x_n & \text{if } 0 \leq x_n \leq \varepsilon\theta, \\ B_{kj}(x_n - \varepsilon\theta) + \varepsilon\theta A_{kj} & \text{if } \varepsilon\theta < x_n \leq \varepsilon, \end{cases} \\ P_{jn}^k &= -P_{nj}^k, \\ P_{ij}^k &= 0 \text{ otherwise.} \end{aligned}$$

It is readily seen that $U = \mathcal{L}(P)$. Moreover P is piecewise linear and $P = 0$ at $x_n = 0$ and $x_n = \varepsilon$, but P does not vanish on the whole boundary of Ω_ε . In order to find the sought function

G , we first remark that, for each $k = 1, \dots, m$, the function P_n^k is proportional to $A_k - B_k$ and compute $\langle P_n^k, A_k - B_k \rangle$:

$$\langle P_n^k, A_k - B_k \rangle = \begin{cases} |A_k - B_k|^2(1 - \theta)x_n & \text{if } 0 \leq x_n \leq \varepsilon\theta, \\ |A_k - B_k|^2\theta(\varepsilon - x_n) & \text{if } \varepsilon\theta < x_n \leq \varepsilon. \end{cases}$$

Note that $\langle P_n^k, A_k - B_k \rangle \geq 0$ in Ω_ε . For each $k = 1, \dots, m$, we introduce the function

$$Q_n^k(x) := -\varepsilon\theta(1 - \theta)(|x_1| + \dots + |x_{n-1}|)(A_k - B_k)$$

and set

$$(3.1) \quad \tilde{P}_n^k := P_n^k + Q_n^k.$$

The function \tilde{P}_n^k is piecewise linear and satisfies $\langle \tilde{P}_n^k, A_k - B_k \rangle \leq 0$ on $\partial\Omega_\varepsilon$. On the other hand $\langle \tilde{P}_n^k, A_k - B_k \rangle > 0$ in a neighborhood of the segment $\{x \in \Omega_\varepsilon : x_1 = \dots = x_{n-1} = 0\}$. Set

$$\tilde{\Omega}_\varepsilon := \{x \in \Omega_\varepsilon : \langle \tilde{P}_n^k, A_k - B_k \rangle > 0\},$$

and define $\tilde{U} := \mathcal{L}(\tilde{P})$, where $\tilde{P} \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m$ is defined by (3.1) and

$$\begin{aligned} \tilde{P}_{jn}^k &= -\tilde{P}_{nj}^k, \\ \tilde{P}_{ij}^k &= 0 \quad \text{otherwise.} \end{aligned}$$

Then

$$\begin{aligned} \tilde{P} &\in (W^{1,\infty}(\tilde{\Omega}_\varepsilon; \mathbb{M}^{n \times n}))^m \text{ is piecewise linear,} \\ \tilde{P}|_{\partial\tilde{\Omega}_\varepsilon} &= 0, \\ \|\tilde{P}\|_{L^\infty(\tilde{\Omega}_\varepsilon)} &< \varepsilon\theta(1 - \theta)|A - B|, \\ \text{dist}(\tilde{U}, \{A, B\}) &< \varepsilon\theta(1 - \theta)|A - B|. \end{aligned}$$

By the Vitali covering theorem one can exhaust Ω by disjoint scaled copies of $\tilde{\Omega}_\varepsilon$. More precisely, there exist $r_i \in (0, 1)$ and $x_i \in \Omega$ such that the sets $\tilde{\Omega}_\varepsilon^i := x_i + r_i\tilde{\Omega}_\varepsilon$ are mutually disjoint and $\text{meas}(\Omega \setminus \cup \tilde{\Omega}_\varepsilon^i) = 0$. Then we define

$$G(x) := \begin{cases} r_i \tilde{P}(r_i^{-1}(x - x_i)) & \text{if } x \in \tilde{\Omega}_\varepsilon^i, \\ 0 & \text{elsewhere.} \end{cases}$$

Finally we set $V := \mathcal{L}(G)$. By choosing ε sufficiently small, it can be easily checked that V satisfies all the required properties. □

Next we study the problem of finding a divergence free field taking values in an open set K and with a prescribed average F . From Lemma 3.3 we know that such problem can be solved provided that $F = \theta A + (1 - \theta)B$ for some $\theta \in (0, 1)$ and $A, B \in K$, with $\text{rank}(A - B) \leq n - 1$. We will see that this procedure can be iterated. More precisely, if $\text{rank}(F - F') \leq n - 1$, and $F' = \theta' A' + (1 - \theta')B'$ for some $\theta' \in (0, 1)$ and $A', B' \in K$, with $\text{rank}(A' - B') \leq n - 1$, then the above problem can be solved also for $\mu F + (1 - \mu)F'$ for all $\mu \in (0, 1)$. This motivates the following definition.

Definition 3.4. We say that $K \subset \mathbb{M}^{m \times n}$ is stable under lamination (or *lamination convex*) if for all $A, B \in K$ such that $\text{rank}(A - B) \leq n - 1$, and all $\theta \in (0, 1)$, one has $\theta A + (1 - \theta)B \in K$. The *lamination convex hull* K^L is defined as the smallest lamination convex set that contains K .

Remark 3.5. It can be easily checked that the lamination convex hull K^L is obtained by successively adding rank- $(n - 1)$ segments, *i.e.*,

$$K^L = \bigcup_i K^i,$$

where $K^0 = K$ and

$$K^i := K^{i-1} \cup \{C : \exists A, B \in K^{i-1}, \theta \in (0, 1) \text{ such that } C = \theta A + (1 - \theta)B, \text{rank}(A - B) \leq n - 1\}.$$

Moreover, if K is open, then all the sets K^i are open.

Lemma 3.6. Suppose that $K \subset \mathbb{M}^{m \times n}$ is open and bounded and that $F \in L^\infty(\Omega; \mathbb{M}^{m \times n})$ is a piecewise constant function which satisfies

$$\begin{aligned} \text{Div} F &= 0 \text{ in } \mathcal{D}'(\Omega; \mathbb{R}^m), \\ F &\in K^L \text{ a.e.} \end{aligned}$$

Then, for each $\delta > 0$, there exists $V_\delta \in L^\infty(\Omega; \mathbb{M}^{m \times n})$ such that

$$\begin{aligned} V_\delta &= \mathcal{L}(G_\delta) + F \text{ with } G_\delta \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m \text{ and piecewise linear,} \\ V_\delta &\in K \text{ a.e.}, \\ \|G_\delta\|_{L^\infty(\Omega)} &< \delta, \\ G_\delta|_{\partial\Omega} &= 0. \end{aligned}$$

Proof. We first assume that F is constant. Then $F \in K^i$ for some i . We argue by induction on i . If $i = 1$, then the result holds by Lemma 3.3. Now assume that the result is true for all $i \leq j$ and let $F \in K^{j+1}$. Then there exist $A, B \in K^j$ such that $\text{rank}(A - B) \leq n - 1$ and $F := \theta A + (1 - \theta)B$ for some $\theta \in (0, 1)$. By Lemma 3.3 there exists a piecewise linear function G such that $\|G\|_{L^\infty(\Omega)} < \delta/2$, $G|_{\partial\Omega} = 0$ and $\text{dist}(\mathcal{L}(G), \{A - F, B - F\}) < \delta$. Since the set K^j is open (see Remark 3.5), for sufficiently small δ , the function $U := \mathcal{L}(G) + F$ satisfies $U \in K^j$ a.e. The latter inclusion implies that U can be written in the form $U = \sum_h \chi_{\Omega_h} (C_h + F)$,

with $C_h + F \in K^j$ and χ_{Ω_h} characteristic functions of disjoint measurable subsets of Ω . We can now apply the induction hypothesis on each subset Ω_h to deduce the existence of functions $G_h \in (W^{1,\infty}(\Omega_h; \mathbb{M}^{n \times n}))^m$ such that

$$\begin{aligned} \mathcal{L}(G_h) + C_h + F &\in K \text{ a.e. in } \Omega_h, \\ \|G_h\|_{L^\infty(\Omega_h)} &< \delta/2, \\ G_h|_{\partial\Omega_h} &= 0. \end{aligned}$$

Finally let $G_\delta(x) := \sum_h \chi_{\Omega_h} G_h + G$ and remark that $\|G_\delta\|_{L^\infty(\Omega)} < \delta$ and $G_\delta|_{\partial\Omega} = 0$.

Now let F be piecewise constant. Then $F = \sum_k \chi_{\Omega_k} F_k$ with $F_k \in K^L$. We now use the previous argument in each subdomain Ω_k where F is constant to obtain the existence of piecewise linear functions $G_\delta^k \in (W^{1,\infty}(\Omega_k; \mathbb{M}^{n \times n}))^m$ such that

$$\begin{aligned} \mathcal{L}(G_\delta^k) + F_k &\in K \text{ a.e.}, \\ \|G_\delta^k\|_{L^\infty(\Omega_k)} &< \delta, \\ G_\delta^k|_{\partial\Omega_k} &= 0. \end{aligned}$$

Finally we define $G_\delta := \sum_k \chi_{\Omega_k} G_\delta^k$ and set $V_\delta := \mathcal{L}(G_\delta) + F$. □

The next step is to pass from open sets to more general sets $K \subset \mathbb{M}^{m \times n}$. In order to do this we approximate K by open sets \mathcal{U}_i and we construct approximate solutions V_i that satisfy $V_i \in \mathcal{U}_i$. Each of the approximate solutions V_{i+1} is obtained from V_i by an application of Lemma 3.6. This suggests in which sense the sets \mathcal{U}_i have to approximate K .

Definition 3.7. Let $K \subset \mathbb{M}^{m \times n}$. We say that a sequence of open sets $\{\mathcal{U}_i\} \subset \mathbb{M}^{m \times n}$ is an *in-approximation* of K if the following three conditions hold:

1. $\mathcal{U}_i \subset \mathcal{U}_{i+1}^L$;
2. the sets \mathcal{U}_i are uniformly bounded;
3. if a sequence $F_i \in \mathcal{U}_i$ converges to F as $i \rightarrow \infty$, then $F \in K$.

Note that a necessary condition for K to admit an in-approximation is that the set $\text{Int}(K^L)$, is non-empty.

We are now ready to state the main result of this section.

Theorem 3.8. Assume that K admits an in-approximation by open sets \mathcal{U}_i and let $F \in \mathcal{U}_1$. Then, for each $\delta > 0$, there exists $V_\delta \in L^\infty(\Omega; \mathbb{M}^{m \times n})$ such that

$$(3.2) \quad V_\delta = \mathcal{L}(H_\delta) + F \text{ with } H_\delta \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m,$$

$$(3.3) \quad V_\delta \in K \text{ a.e.},$$

$$(3.4) \quad \|H_\delta\|_{L^\infty(\Omega)} < \delta,$$

$$(3.5) \quad H_\delta|_{\partial\Omega} = 0.$$

Proof. We construct a sequence of piecewise constant divergence free maps V_i such that

$$(3.6) \quad V_i = \mathcal{L}(H_i) + F \text{ with } H_i \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m,$$

$$V_i \in \mathcal{U}_i \text{ a.e.},$$

$$\|H_{i+1} - H_i\|_{L^\infty(\Omega)} < \delta_{i+1}$$

$$H_i|_{\partial\Omega} = 0.$$

To start with, set $H_1 := 0$ and $V_1 := F$. Since $F \in \mathcal{U}_2^L$, we can apply Lemma 3.6 to deduce the existence of a function V_2 such that

$$\begin{aligned}
V_2 &= \mathcal{L}(G_2) + F \text{ with } G_2 \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m \text{ and piecewise linear,} \\
V_2 &\in \mathcal{U}_2 \quad \text{a.e. ,} \\
\|G_2\|_{L^\infty(\Omega)} &< \delta_2, \\
G_2|_{\partial\Omega} &= 0.
\end{aligned}$$

with $\delta_2 = \delta$. We then define $H_2 = G_2$. To construct V_{i+1} and δ_{i+1} from V_i and δ_i , we proceed as follows. Let

$$\Omega_i := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/2^i\}.$$

Let ϱ be a standard smooth convolution kernel in \mathbb{R}^n , i.e., $\rho \geq 0$, $\int \rho = 1$, $\text{Spt } \rho \subset \{|x| < 1\}$, and let $\varrho_{\varepsilon_i}(x) := \varepsilon_i^{-n} \varrho(x/\varepsilon_i)$. We choose $\varepsilon_i \in (0, 2^{-i})$ so that

$$(3.7) \quad \|\varrho_{\varepsilon_i} * \mathcal{L}(H_i) - \mathcal{L}(H_i)\|_{L^1(\Omega_i)} < \frac{1}{2^i}.$$

where the convolution acts on each entry of the matrix field $\mathcal{L}(H_i)$. Now let

$$(3.8) \quad \delta_{i+1} = \delta_i \varepsilon_i.$$

and use Lemma 3.6 to construct a function $G_{i+1} \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m$ such that

$$\begin{aligned}
\mathcal{L}(G_{i+1}) + V_i &\in \mathcal{U}_{i+1} \quad \text{a.e. ,} \\
\|G_{i+1}\|_{L^\infty(\Omega)} &< \delta_{i+1}.
\end{aligned}$$

Next we set $H_{i+1} := \sum_{j=2}^{i+1} G_j$ and define V_{i+1} according to (3.6), so that

$$V_{i+1} = \mathcal{L}(G_{i+1}) + V_i.$$

Since $\sum_{i=2}^{\infty} \delta_i < \delta/2$ and, for $i > j$,

$$(3.9) \quad \|H_i - H_j\|_{L^\infty(\Omega)} \leq \sum_{k=j+1}^i \|G_k\|_{L^\infty(\Omega)},$$

we find that $H_i \rightarrow H_\infty$ uniformly. Moreover, since by construction the sequence $\{H_i\}$ is uniformly bounded in $W^{1,\infty}(\Omega)$, we have that

$$\mathcal{L}(H_i) \xrightarrow{*} \mathcal{L}(H_\infty) \text{ in } L^\infty \text{ weak } *.$$

Taking $H_\delta = H_\infty$ and $V_\delta := \mathcal{L}(H_\delta) + F$, we see that conditions (3.2)-(3.4)-(3.5) hold. We are left to show that $V_\delta \in K$ a.e. To this end, we will prove the strong convergence of $\mathcal{L}(H_i)$ to $\mathcal{L}(H_\delta)$ in L^1 . Indeed, since

$$\int_{\Omega} (\mathcal{L}(\Phi)(y))_{kj} \varrho(x-y) dy = - \sum_{i=1}^n \int_{\Omega} \Phi_{ij}^k(y) \frac{\partial \varrho}{\partial x_i}(x-y) dy, \quad \forall \Phi \in (W^{1,\infty}(\Omega; \mathbb{M}^{n \times n}))^m,$$

and since $\|\nabla \varrho_{\varepsilon_i}\|_{L^1} < C/\varepsilon_i$, we deduce from (3.8) and (3.9)

$$\begin{aligned}
\|\varrho_{\varepsilon_i} * (\mathcal{L}(H_i) - \mathcal{L}(H_\infty))\|_{L^1(\Omega_i)} &\leq \frac{C}{\varepsilon_i} \|H_i - H_\infty\|_{L^\infty(\Omega)} \\
&\leq \frac{C}{\varepsilon_i} \sum_{k=i+1}^{\infty} \delta_k \\
&\leq 2 \frac{C}{\varepsilon_i} \delta_{i+1} \\
(3.10) \qquad \qquad \qquad &\leq C' \delta_i.
\end{aligned}$$

Combining (3.7) and (3.10) yields

$$\begin{aligned}
\|\mathcal{L}(H_i) - \mathcal{L}(H_\infty)\|_{L^1(\Omega)} &\leq C' \delta_i + 2^{-i} + \|\varrho_{\varepsilon_i} * \mathcal{L}(H_\infty) - \mathcal{L}(H_\infty)\|_{L^1(\Omega_i)} \\
&\quad + \|\mathcal{L}(H_i) - \mathcal{L}(H_\infty)\|_{L^1(\Omega \setminus \Omega_i)}.
\end{aligned}$$

Since $\mathcal{L}(H_i)$ and $\mathcal{L}(H_\infty)$ are bounded, we obtain $\mathcal{L}(H_i) \rightarrow \mathcal{L}(H_\infty)$ in $L^1(\Omega)$ and thus $V_i \rightarrow V_\delta$ in $L^1(\Omega)$. Therefore there exists a subsequence V_{i_j} such that

$$V_{i_j} \rightarrow V_\delta \quad \text{a.e.}$$

It follows from the definition of in-approximation that

$$V_\delta \in K \quad \text{a.e.}$$

□

4. APPLICATIONS OF THE CONVEX INTEGRATION RESULTS TO THE STUDY OF THE BORN-INFELD EQUATIONS

4.1. Approach by Young measures. We formulate problem (1.5)-(1.7) in the language of \mathcal{A} -convexity (see, *e.g.*, [4], [13]). Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain and let \mathcal{M} be defined by (2.1). Let $A^{(1)}, A^{(2)}, A^{(3)} \in \mathbb{M}^{2 \times 10}$ be defined as follows

$$\begin{aligned}
A^{(1)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
A^{(2)} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
A^{(3)} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

We introduce the operators

$$\begin{aligned}
\mathcal{A}(V) &:= \sum_{i=1}^3 A^{(i)} \frac{\partial V}{\partial x_i}, \quad V : \Omega \rightarrow \mathbb{R}^{10}, \\
\mathbb{A}(w) &:= \sum_{i=1}^3 A^{(i)} w_i \in \text{Lin}(\mathbb{R}^{10}; \mathbb{R}^2), \quad w \in \mathbb{R}^3,
\end{aligned}$$

where $\text{Lin}(\mathbb{R}^{10}; \mathbb{R}^2)$ denotes the space of linear operators from \mathbb{R}^{10} to \mathbb{R}^2 . The operator \mathcal{A} satisfies the *constant rank* property, *i.e.*,

$$\text{rank} \mathbb{A}(w) = 2 \quad \forall w \in \mathbb{S}^2,$$

where \mathbb{S}^2 is the unit sphere in \mathbb{R}^3 . Moreover

$$\ker \mathbb{A}(w) = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^4 : \alpha \perp w, \beta \perp w\} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^4.$$

Therefore the *characteristic cone* Λ is all of \mathbb{R}^{10} . Indeed

$$\Lambda := \cup_{w \in \mathbb{S}^2} \ker \mathbb{A}(w) = \{(\alpha, \beta) \in \mathbb{R}^3 \times \mathbb{R}^3 : \exists \xi \in \mathbb{R}^3 \text{ such that } \xi \perp \alpha, \xi \perp \beta\} \times \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^4.$$

Thus Λ -convexity reduces to standard convexity. Conditions (1.5)-(1.6) can be rewritten in terms of the constant rank operator \mathcal{A} as

$$(4.1) \quad \mathcal{A}(V_j) = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

$$(4.2) \quad V_j \in \mathcal{M} \quad \text{a.e. in } \Omega.$$

One can also consider the approximate version of (4.1), where the differential constraint on the sequence $\{V_j\}$ is replaced by the weaker condition

$$(4.3) \quad \mathcal{A}(V_j) \rightarrow 0 \quad \text{strongly in } W^{-1,p'}(\Omega).$$

The next Theorems 4.1 and 4.3 and their corollaries are special case of more general results contained in [4], where more general constant-rank operators are considered. Let us also mention that, in the gradient case, *i.e.*, when the operator \mathcal{A} is the curl operator, such results were first established by Kinderlehrer and Pedregal [7].

Theorem 4.1. *Let $1 \leq p < +\infty$. Suppose that the sequence $\{V_j\}$ generates the Young measure $\{\nu_x\}_{x \in \Omega}$ and let $V_j \rightharpoonup V$ in $L^p(\Omega; \mathbb{R}^{10})$. If $\{V_j\}$ satisfies (4.1), or its approximate version (4.3), then*

$$\begin{aligned} \langle \nu_x, id \rangle &= V(x) \in \ker \mathcal{A}, \\ \int_{\Omega} \int_{\mathbb{R}^{10}} |M|^p d\nu_x(M) &< \infty. \end{aligned}$$

If in addition the sequence $\{V_j\}$ is uniformly bounded in $L^\infty(\Omega; \mathbb{R}^{10})$ and (4.2) holds, then

$$(4.4) \quad \text{supp } \nu_x \subset \mathcal{M} \quad \text{for a.e. } x \in \Omega.$$

Corollary 4.2. *Under the assumptions of Theorem 4.1, if (4.4) holds, then*

$$V(x) \in \mathcal{M}^c \quad \text{for a.e. } x \in \Omega.$$

Theorem 4.3. *Let $1 \leq p < +\infty$, and let $\{\nu_x\}_{x \in \Omega}$ be a weakly measurable family of probability measures on \mathbb{R}^{10} . Suppose that*

$$\begin{aligned} \langle \nu_x, id \rangle &\in \ker \mathcal{A}, \\ \int_{\Omega} \int_{\mathbb{R}^{10}} |M|^p d\nu_x(M) &< \infty. \end{aligned}$$

Then there exists a sequence $\{V_j\} \subset L^p(\Omega; \mathbb{R}^{10})$ satisfying (4.1) that generates $\{\nu_x\}$.

Corollary 4.4. *Let $V \in L^p(\Omega; \mathbb{R}^{10})$. Suppose that $\mathcal{A}(V) = 0$ and $V \in \mathcal{M}^c$ a.e. Then there exists a sequence $\{V_j\} \subset L^p(\Omega; \mathbb{R}^{10})$ satisfying (4.1) such that*

$$\text{dist}(V_j, \mathcal{M}) \rightarrow 0 \text{ in } L^p(\Omega) \text{ and } V_j \rightharpoonup V \text{ in } L^p(\Omega; \mathbb{R}^{10}).$$

Remark 4.5. By suitably projecting the sequence $\{V_j\}$ provided by Corollary 4.4 onto \mathcal{M} , one can obtain a sequence $\{\tilde{V}_j\} \subset L^p(\Omega; \mathbb{R}^{10})$ satisfying (4.3) such that

$$\tilde{V}_j \in \mathcal{M} \text{ a.e. and } \tilde{V}_j \rightharpoonup \mathcal{M} \text{ in } L^p(\Omega).$$

4.2. Approach by convex integration. We wish to use the convex integration approach developed in Section 3 to find maps which satisfy both the constraints (4.1)-(4.2) exactly and have a prescribed average in the convex hull \mathcal{M}^c . First, we introduce the appropriate definition of in-approximation for the set \mathcal{M} .

Definition 4.6. *We say that a sequence of open sets $\{\mathcal{U}_i\} \subset \mathbb{R}^{10}$ is an in-approximation of \mathcal{M} if the following three conditions hold:*

1. $\mathcal{U}_i \subset \mathcal{U}_{i+1}^c$;
2. the sets \mathcal{U}_i are uniformly bounded;
3. if a sequence $F_i \in \mathcal{U}_i$ converges to F as $i \rightarrow \infty$, then $F \in \mathcal{M}$.

We will need the following characterization of \mathcal{M}^c , which is due to Serre [12].

Theorem 4.7. *The convex hull \mathcal{M}^c is defined by*

$$\mathcal{M}^c = \{(B, D, P, h) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : h \geq \sqrt{1 + |D|^2 + |B|^2 + |P|^2 + 2|P - D \wedge B|}\}.$$

Proposition 4.8. *The set \mathcal{M} admits an in-approximation.*

Proof. We introduce the sequence of sets

$$(4.5) \quad \mathcal{U}_i := \{(D, B, P, h) \in \mathbb{R}^{10} : |P - D \wedge B| < \sigma_i, \\ \sqrt{1 + |D|^2 + |B|^2 + |P|^2} + \varepsilon_i < h < \sqrt{1 + |D|^2 + |B|^2 + |P|^2} + \varepsilon_{i-1}\},$$

where $\{\sigma_i\}$ and $\{\varepsilon_i\}$ are decreasing sequences of real numbers such that $\varepsilon_i \searrow 0$ and $\sigma_i \searrow 0$. Next we check that if

$$(4.6) \quad 2\sigma_i < (\varepsilon_i - \varepsilon_{i+1})^2,$$

then

$$\mathcal{U}_i \subset \mathcal{U}_{i+1}^c, \quad \forall i.$$

Indeed we have

$$\mathcal{U}_{i+1}^c \supset \{h > \sqrt{1 + |D|^2 + |B|^2 + |P|^2 + 2|P - D \wedge B|} + \varepsilon_{i+1}\},$$

and, if $(D, B, P, h) \in \mathcal{U}_i$, then

$$h > \sqrt{1 + |D|^2 + |B|^2 + |P|^2} + \varepsilon_i > \sqrt{1 + |D|^2 + |B|^2 + |P|^2 + 2|P - D \wedge B|} + \varepsilon_{i+1}.$$

The second inequality above follows from

$$\begin{aligned} \sqrt{1 + |D|^2 + |B|^2 + |P|^2} + \varepsilon_i - \varepsilon_{i+1} &> \sqrt{1 + |D|^2 + |B|^2 + |P|^2 + (\varepsilon_i - \varepsilon_{i+1})^2} \\ &> \sqrt{1 + |D|^2 + |B|^2 + |P|^2 + 2\varepsilon_i} \\ &> \sqrt{1 + |D|^2 + |B|^2 + |P|^2 + 2|P - D \wedge B|}. \end{aligned}$$

Then, for any given $m_1 > 0$, one can find a bounded increasing sequence of numbers $\{m_i\}$ such that the sets

$$(4.7) \quad \hat{\mathcal{U}}_i := \mathcal{U}_i \cap \{(D, B, P, h) \in \mathbb{R}^{10} : |D| < m_i, |B| < m_i\}$$

define an in-approximation of K . □

Lemma 4.9. *Let $F \in \text{Int}(\mathcal{M}^c)$. Then there exists an in-approximation $\{\hat{\mathcal{U}}_i\}$ of the form (4.7) such that $F \in \hat{\mathcal{U}}_1$.*

Proof. We first choose $m_1, \sigma_1, \varepsilon_1$ and ε_2 so that

$$\begin{aligned} |F_D| &< m_1, \quad |F_B| < m_1, \quad |F_P - F_D \wedge F_B| < \sigma_1, \\ F_h &< \sqrt{1 + |F_D|^2 + |F_B|^2 + |F_P|^2} + \varepsilon_1, \\ \sqrt{1 + |F_D|^2 + |F_B|^2 + |F_P|^2} + \varepsilon_2 &< \sqrt{1 + |F_D|^2 + |F_B|^2 + |F_P|^2 + 2|F_P - F_D \wedge F_B|}, \\ 2\sigma_1 &< (\varepsilon_1 - \varepsilon_2)^2. \end{aligned}$$

We then complete the sequences $\{\sigma_i\}, \{\varepsilon_i\}$ in order to fulfill (4.6), and $\{m_i\}$ accordingly. □

A straightforward adaptation of Lemma 3.3 and Lemma 3.6 to the \mathcal{A} -setting yields the following two results.

Lemma 4.10. *Let $M, N \in \mathbb{R}^{10}$ and let $F := \theta M + (1 - \theta)N$ for some $\theta \in (0, 1)$. Then for each $\delta > 0$, there exists $V \in L^\infty(\Omega; \mathbb{R}^{10})$ such that*

$$(4.8) \quad V_D = \text{curl } G_1 + F_D, \quad V_B = \text{curl } G_2 + F_B \quad \text{with } G_1, G_2 \in W^{1,\infty}(\Omega; \mathbb{R}^3) \\ \text{and piecewise linear,}$$

$$(4.9) \quad \|G_1\|_{L^\infty(\Omega)} < \delta, \quad \|G_2\|_{L^\infty(\Omega)} < \delta,$$

$$(4.10) \quad G_1|_{\partial\Omega} = 0, \quad G_2|_{\partial\Omega} = 0,$$

$$(4.11) \quad \text{dist}((V_D, V_B), \{(M_D, M_B), (N_D, N_B)\}) < \delta,$$

$$(4.12) \quad (V_P, V_h) \in \{(M_P, M_h), (N_P, N_h)\} \text{ a.e.},$$

$$(4.13) \quad \int_{\Omega} V \, dx = F.$$

Proof. As in the proof of Lemma 3.3 we define the function $U := \chi M + (1 - \chi)N$ on the domain Ω_ε so that $\mathcal{A}U = 0$ and $\int_{\Omega_\varepsilon} U = F$. In order to define V , we follow the proof of Lemma 3.3 and we modify U_D and U_B accordingly, so that (4.8)-(4.11) are satisfied. Leaving the other components of U unchanged yields (4.12)-(4.13). □

Lemma 4.11. *Let $\{\hat{\mathcal{U}}_i\}$ be the in-approximation of \mathcal{M} provided by (4.7). Suppose that $F \in L^\infty(\Omega; \mathbb{R}^{10})$ is a piecewise constant function which satisfies*

$$\begin{aligned} \mathcal{A}F &= 0 \text{ in } \mathcal{D}'(\Omega), \\ F &\in \hat{\mathcal{U}}_i^c \text{ a.e. for some } i \geq 1. \end{aligned}$$

Then, for each $\delta > 0$, there exists $V_\delta \in L^\infty(\Omega; \mathbb{R}^{10})$ such that

$$\begin{aligned} (V_\delta)_D &= \text{curl } G_1 + F_D, \quad (V_\delta)_B = \text{curl } G_2 + F_B \quad \text{with } G_1, G_2 \in W^{1,\infty}(\Omega; \mathbb{R}^3) \text{ and piecewise linear,} \\ \|G_1\|_{L^\infty(\Omega)} &< \delta, \quad \|G_2\|_{L^\infty(\Omega)} < \delta, \\ G_1|_{\partial\Omega} &= 0, \quad G_2|_{\partial\Omega} = 0, \\ V_\delta &\in \hat{\mathcal{U}}_i \text{ a.e.}, \\ \int_{\Omega} V_\delta dx &= F. \end{aligned}$$

Proof. The proof is done by induction like that of Lemma 3.6, with an iterative use of the basic construction provided by Lemma 4.10. \square

Theorem 4.12. *Let $F \in \text{Int}(\mathcal{M}^c)$. Then for each $\delta > 0$ there exists $V_\delta \in L^\infty(\Omega; \mathbb{R}^{10})$ such that*

$$\begin{aligned} (V_\delta)_D &= \text{curl } H_1 + F_D, \quad (V_\delta)_B = \text{curl } H_2 + F_B, \quad \text{with } H_1, H_2 \in W^{1,\infty}(\Omega; \mathbb{R}^3), \\ \|H_1\|_{L^\infty(\Omega)} &< \delta, \quad \|H_2\|_{L^\infty(\Omega)} < \delta, \\ H_1|_{\partial\Omega} &= 0, \quad H_2|_{\partial\Omega} = 0, \\ V &\in \mathcal{M} \text{ a.e.}, \\ \int_{\Omega} V dx &= F. \end{aligned}$$

Proof. By Lemma 4.9 there exists an in-approximation $\{\hat{\mathcal{U}}_i\}$ of the form (4.5) with $F \in \hat{\mathcal{U}}_1$. Following the strategy of the proof of Theorem 3.8, we construct a sequence of functions $V_i : \Omega \rightarrow \mathbb{R}^{10}$ such that

$$\begin{aligned} V_i &\in \hat{\mathcal{U}}_i \text{ a.e.}, \\ \mathcal{A}V_i &= 0 \text{ in } \mathcal{D}'(\Omega), \\ \int_{\Omega} V_i dx &= F, \\ V_i &\rightarrow V \text{ in } L^1(\Omega), \end{aligned}$$

where the potentials of $(V_i)_D - F_D$ and $(V_i)_B - F_B$ can be chosen to be smaller than δ and with zero boundary data. This can be done as in the proof of Theorem 3.8 by iteratively applying Lemma 4.11. Thus one gets $V_i \xrightarrow{*} V$ in $L^\infty(\Omega)$ weak *, and $(V_i)_D \rightarrow V_D, (V_i)_B \rightarrow V_B$ strongly in $L^1(\Omega)$. Then, since the sequence $\{V_i\}$ is uniformly bounded in $L^\infty(\Omega)$, one has

$$(V_i)_D \wedge (V_i)_B \rightarrow V_D \wedge V_B \text{ strongly in } L^1(\Omega).$$

Moreover, since $V_i \in \hat{\mathcal{U}}_i$ a.e., recalling the definition of $\hat{\mathcal{U}}_i$ we find

$$\|(V_i)_P - ((V_i)_D \wedge (V_i)_B)\|_{L^\infty(\Omega)} < \sigma_i,$$

and therefore

$$(V_i)_P \rightarrow V_D \wedge V_B \text{ strongly in } L^1(\Omega).$$

By definition of $\hat{\mathcal{U}}_i$, we also have that

$$\left\| (V_i)_h - \sqrt{1 + |(V_i)_D|^2 + |(V_i)_B|^2 + |(V_i)_h|^2} \right\|_{L^\infty(\Omega)} < \varepsilon_{i-1},$$

and therefore $(V_i)_h \rightarrow V_h$ strongly in $L^1(\Omega)$.

□

Remark 4.13. Theorem 1.1 follows as a corollary of Theorem 4.12.

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